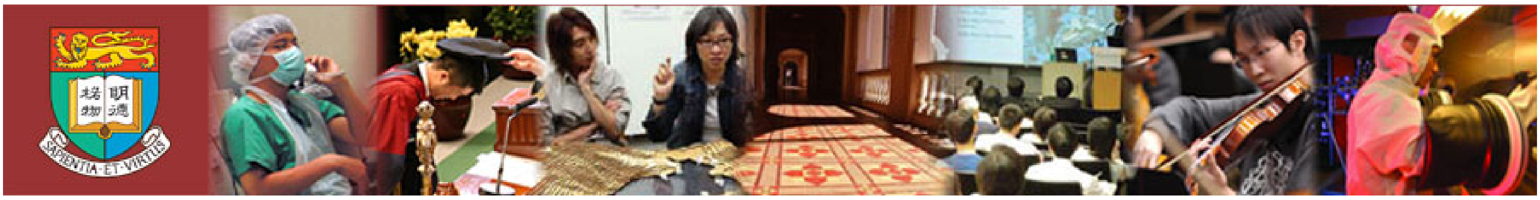

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Fusion Estimation for Two Sensors with Nonuniform Estimation Rates

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Abstract— **The fusion estimation is investigated in this paper for two-sensor discrete-time stochastic systems. A finite-horizon optimal linear estimator is designed for each sensor to generate local estimates with a nonuniform estimation rate. Then, a fusion rule with matrix weights in the linear minimum variance sense is designed for each sensor to fuse local estimates from itself and the other sensors. The proposed algorithm reduces to the one that can be used to design asynchronous fusion estimators with uncorrelated measurement noises. Finally, the effectiveness of the proposed results is illustrated by a simulation example of a maneuvering target tracking system.**

I. INTRODUCTION

As one of the important issues in information fusion, the information fusion estimation has attracted considerable research interest during the past decades, and has found applications in a variety of areas such as integrated navigation systems for tracking targets [1]. Many useful fusion estimation methods have been presented in the literature, such as the state-vector fusion estimation and measurement fusion estimation [2], centralized fusion estimation where all measurements are transmitted to a fusion center for processing [3] and distributed fusion estimation where the information from local estimators are collected to yield global optimal or suboptimal state estimate according to certain fusion criterion [4]. In the conventional fusion estimation, it is implicitly assumed that the measurements are sampled uniformly, and thus the estimates are generated periodically with a single rate. However, in practical applications, one often encounters situations where the sensors temporarily fail to provide useful measurements or the measurements may be lost during transmission in network environments, etc. [5], [6]. That is to say, the measurements may not be available to the sensors at certain sampling instants, thus the estimation has to be performed with nonuniform rates, as illustrated in (a) of Fig. 1.

Some related results on the estimation with temporarily unavailable measurements can be found in the literature about networked estimation with packet losses [7]. In these results, it is usually assumed that the estimator input keeps the last available value or is set to zero if the current measurement is lost, and the estimates are actually generated periodically with a uniform rate. Recently, a stochastic sampling method was presented in [8] to design sampled-data H_∞ filters with a nonuniform filtering rate taking only two values according to a known probability distribution law. However, the results in [8] is concerned with the single sensor estimation problem.

There are mainly two difficulties in designing fusion estimators with nonuniform estimation rates. The first difficulty is how to design each local estimator with nonuniform estimation rates. The second difficulty is how to fuse the local estimates generated asynchronously by the sensors, which is also due to the nonuniform estimation. Taking a particular sensor in the estimation system for example, not all the local estimates may be available for fusing at a particular estimating instant, and the number of available local estimates for fusing is time-varying at different estimating instants. The difficulty can be immediately solved by applying some distributed fusion estimation algorithms ([4], [9]) by assuming that all the measurement noises are mutually uncorrelated, such as the results presented in [10]. For the case where the measurement noises are correlated, the asynchronous fusion estimation with nonuniform estimation rates is much more complex and remains to be unsolved.

This paper presents a design method for the fusion estimators with nonuniform estimation rates by using the lifting technique and a distributed fusion criterion with matrix weights in the linear minimum variance sense. For ease of presenting the main idea, it is considered that there are two sensors and the estimation rate in each sensor takes only two values. Assuming that the estimation rate in each sensor varies according to a white Bernoulli sequence, each local estimation system is modeled as a discrete stochastic system with a stochastic parameter, and optimal linear estimators are designed by using projection principle and innovation analysis. Then, a fusion rule is designed for each sensor by using the distributed fusion criterion with matrix weights. Each sensor generates fused estimates according to the designed fusion rule if local estimates from the other sensors are available, and keeps its own estimates as the fused ones otherwise.

II. PROBLEM STATEMENT AND MODELING

A. Problem Statement

Consider a linear discrete stochastic system described by the following state-space model

$$x(T_{k+1}) = Ax(T_k) + B\omega(T_k),\ k = 0, 1, 2, \dots \qquad (1)$$

where $x(T_k) \in \Re^n$ is the system state, $\omega(T_k) \in \Re^{q_\omega}$ is a zero mean white noise with variance Q_ω, i.e., $\mathbf{E}\{\omega(T_i)\omega^{\mathrm{T}}(T_j)\} = Q_\omega\delta_{ij}$. The sampling period is denoted by h, and $h = T_{k+1}-$

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T_k, $k = 0, 1, 2, \ldots$. The outputs of system (1) are measured by two sensors, say, sensor r and sensor s, and the output equations are given by

$$y_i(T_k) = C_i x(T_k) + D_i \upsilon_i(T_k),\ i \in Z_0 = \{r, s\} \qquad (2)$$

where $y_i(T_k) \in \Re^{p_i}$, and $\upsilon_i(T_k) \in \Re^{q_i}$ are zero mean white measurement noises with constant variances Q_i, i.e., $\mathbf{E}\{\upsilon_i(T_l)\upsilon_i^{\mathrm{T}}(T_j)\} = Q_i\delta(l-j)$, where $\delta(l-j)$ is the Dirac Delta function.

There is no fusion center in the estimation system, each sensor acts also as an estimator. At each time step, each sensor i first generates local estimates $\hat{x}_i = f_i(y_i)$ by using measurements from itself, and then generates fused estimates $\hat{x}_{oi} = g_i(\hat{x}_1, \hat{x}_2)$ by collecting local estimates from itself and another sensor, where $f_i(.)$ and $g_i(.)$ are the local estimation algorithm and the fusion rule to be designed at sensor i, respectively. As mentioned before, due to some unexpected reasons, such as temporary sensor failures and packet losses (where the measurements are transmitted via networks), the measurements may not be available for generating estimates at certain sampling instants, and the estimation has to be performed at a nonuniform rate, as illustrated in (b) and (c) of Fig. 1. In Fig. 1, $t_{i,k}$, $k = 0, 1, 2, \ldots$ denote the sampling instants at which the measurements are available and the estimates are generated at sensor i. The sampling period of sensor i is $h_i(t_{i,k}) = t_{i,k+1} - t_{i,k}$, which is time-varying and integer multiple of h. Then, the output equations in (2) are represented by

$$y_i(t_{i,k}) = C_i x(t_{i,k}) + D_i \upsilon_i(t_{i,k}) \qquad (3)$$

Suppose that $\forall\ i \in Z_0$, the maximal value of $h_i(t_{i,k})$ is bounded, and $h_i(t_{i,k})$ takes a finite number of m values. Then, $h_i(t_{i,k})$ can be denoted as

$$h_i(t_{i,k}) = a_i(t_{i,k})h \qquad (4)$$

where $a_i(t_{i,k}) \in \{a_1, \ldots, a_m\}$, $\forall\ i \in Z_0,\ k = 0, 1, 2, \ldots$, and a_j, $j = 1, \ldots, m$ are positive integers.

It can be seen from (b) and (c) in Fig. 1 that $t_{i,k}$ is generally not equal $t_{j,k}$, $k = 0, 1, 2, \ldots$. This is to say, the sensors generate local estimates asynchronous. Hence, there are two issues that should be considered in designing the fusion estimators. The first issue is how to design an optimal local estimator for each sensor with a nonuniform estimation rate, and the second issue is how to design an optimal fusion rule for each sensor to fuse estimates asynchronously. In order to present our main idea in a clear and convenient way, we consider the case where $h_i(t_{i,k})$ takes only two values $a_1 h$ and $a_2 h$. The results to be presented can be extended to the general case where $h_i(t_{i,k})$ takes m values by following some similar procedures to be given in the following sections. Denote by P_i and P_{oi} the local estimation error variance and fused estimation error variance of sensor i, respectively. Then, the objective of the paper is given as follows.

Objective of the paper: For system (1) and (3), design an optimal local estimator $f_i(.)$ and an optimal fusion rule $g_i(.)$ with matrix weights for each sensor such that the fused estimates $\hat{x}_{oi}(t_{i,k}|t_{i,k})$ are unbiased optimal estimates of the system state $x(t_{i,k})$, i.e., $\mathbf{E}\{\hat{x}_{oi}(t_{i,k}|t_{i,k})\} = \mathbf{E}\{x(t_{i,k})\}$, and $P_{oi}(t_{i,k}|t_{i,k}) = \min\{P(t_{i,k}|t_{i,k})\}$, $P_{oi}(t_{i,k}|t_{i,k}) \leq P_i(t_{i,k}|t_{i,k})$, where $P(t_{i,k}|t_{i,k})$ denotes the estimating error variance of an arbitrary fusion estimator with matrix weights, $i \in Z_0$.

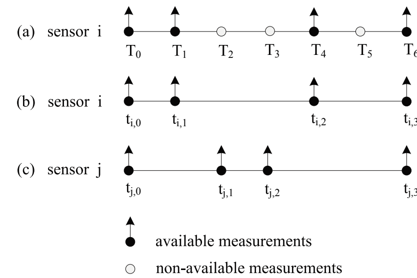

Fig. 1. A timing diagram of asynchronous fusion estimation.

B. Modeling of the Estimation Systems

A system model with time scales $t_{i,k}$ is needed before a local estimator can be designed for each sensor i, $i \in Z_0$, and the model is established in this subsection.

If the measurements from sensor i are available with the period $a_1 h$, then one obtains by applying (1) recursively that

$$\begin{aligned} x_i(t_{i,k+1}) &= A_1 x_i(t_{i,k}) + \omega_1(t_{i,k}) &(5)\\ y_i(t_{i,k}) &= C_i x_i(t_{i,k}) + D_i \upsilon_i(t_{i,k}) &(6) \end{aligned}$$

where $A_1 = A^{a_1}$ and $\omega_1(t_{i,k}) = \sum_{j=0}^{a_1-1} A^{a_1-j-1}B\omega(t_{i,k} + jh)$. Similarly, if the measurements from sensor i are available with the period $a_2 h$, then one has

$$\begin{aligned} x_i(t_{i,k+1}) &= A_2 x_i(t_{i,k}) + \omega_2(t_{i,k}) &(7)\\ y_i(t_{i,k}) &= C_i x_i(t_{i,k}) + D_i \upsilon_i(t_{i,k}) &(8) \end{aligned}$$

where $A_2 = A^{a_2}$ and $\omega_2(t_{i,k}) = \sum_{j=0}^{a_2-1} A^{a_2-j-1}B\omega(t_{i,k} + jh)$. Since $h_i(t_{i,k})$ varies between $a_1 h$ and $a_2 h$, the estimation system model of sensor i with sampling period $h_i(t_{i,k})$ is given by

$$\begin{aligned} x_i(t_{i,k+1}) &= \tilde{A}_i(t_{i,k}) x_i(t_{i,k}) + \omega_i(t_{i,k}) &(9)\\ y_i(t_{i,k}) &= C_i x_i(t_{i,k}) + D_i \upsilon_i(t_{i,k}) &(10) \end{aligned}$$

where $\tilde{A}_i(t_{i,k}) \in \{A_1, A_2\}$ and

$$\begin{aligned} \omega_i(t_{i,k}) &= \sum_{j=0}^{\tilde{a}_i(t_{i,k})-1} A^{\tilde{a}_i(t_{i,k})-j-1}B\omega(t_{i,k} + jh) &(11)\\ \tilde{a}_i(t_{i,k}) &\in \{a_1, a_2\} &(12) \end{aligned}$$

By the expression of $\omega_i(t_{i,k})$, it can be seen that $\omega_i(t_{i,k})$ is zero mean. Besides, since $\forall\ l \neq k$, $t_{i,k} + jh < t_{i,k} + \tilde{a}_i(t_{i,k})h = t_{i,l}$, $\forall\ j = 0, 1, \ldots, \tilde{a}_i(t_{i,k}) - 1$ or $t_{i,l} + jh < t_{i,l} + \tilde{a}_i(t_{i,l})h = t_{i,k}$, $\forall\ j = 0, 1, \ldots, \tilde{a}_i(t_{i,l}) - 1$, one has $\mathbf{E}\{\omega_i(t_{i,k})\omega_i^{\mathrm{T}}(t_{i,l})\} = 0$, $l \neq k$. Moreover, define $Q_{\omega_i}(t_{i,k}) \triangleq \mathbf{E}\{\omega_i(t_{i,k})\omega_i^{\mathrm{T}}(t_{i,k})\}$, then by (11) one has that

$$Q_{\omega_i}(t_{i,k}) = \sum_{j=0}^{\tilde{a}_i(t_{i,k})-1} A^{\tilde{a}_i(t_{i,k})-j-1} B Q_\omega B^{\mathrm{T}} (A^{\tilde{a}_i(t_{i,k})-j-1})^{\mathrm{T}}$$

Thus, $\omega_i(t_{i,k})$ is a zero mean white noise with a time-varying variance $Q_{\omega_i}(t_{i,k})$.

The estimation system at each sensor with a nonuniform estimation rate is finally modeled as a time-varying stochastic system with a white process noise that has a time-varying variance. If the sensor knows exactly the pace of its sampling period $h_i(t_{i,k})$, then one may design a finite-horizon Kalman estimator for each sensor based on model (9) and (10). Otherwise, one is unable to use the model (9) and (10) to design estimators. In this paper, it is assumed that the sensors do not know exactly the pace of the sampling period, but instead, the sensor only knows that $h_i(t_{i,k})$ take values in $\{a_1 h, a_2 h\}$ with some known probabilities. Specifically, it is assumed that $h_i(t_{i,k})$ varies between $a_1 h$ and $a_2 h$ according to a white binary-valued Bernoulli sequence $\alpha_i(t_{i,k})$, and

$$\begin{aligned} &\mathrm{Prob}\{h_i(t_{i,k}) = a_1 h\} = \mathrm{Prob}\{\alpha_i(t_{i,k}) = 1\} \\ &= \mathbf{E}\{\alpha_i(t_{i,k})\} = \alpha_i \end{aligned} \tag{13}$$

$$\begin{aligned} &\mathrm{Prob}\{h_i(t_{i,k}) = a_2 h\} = \mathrm{Prob}\{\alpha_i(t_{i,k}) = 0\} \\ &= 1 - \mathrm{Prob}\{\alpha_i(t_{i,k}) = 1\} = 1 - \alpha_i \end{aligned} \tag{14}$$

where $0 \leq \alpha_i \leq 1$. By (13) and (14), one has

$$\tilde{A}_i(t_{i,k}) = \alpha_i(t_{i,k}) A_1 + (1 - \alpha_i(t_{i,k})) A_2 \tag{15}$$

$$\tilde{a}_i(t_{i,k}) = \alpha_i(t_{i,k}) a_1 + (1 - \alpha_i(t_{i,k})) a_2 \tag{16}$$

Moreover, by (11), (15) and (16) one has that

$$\begin{aligned} Q_{\omega_i} &= \mathbf{E}\{\omega_i(t_{i,k})\omega_i^{\mathrm{T}}(t_{i,k})\} \\ &= (1-\alpha_i) \sum_{j=0}^{a_2-1} A^{a_2-j-1} B Q_\omega B^{\mathrm{T}} (A^{a_2-j-1})^{\mathrm{T}} \\ &\quad + \alpha_i \sum_{j=0}^{a_1-1} A^{a_1-j-1} B Q_\omega B^{\mathrm{T}} (A^{a_1-j-1})^{\mathrm{T}} \end{aligned} \tag{17}$$

In what follows, a fusion estimator will be designed for each sensor based on the system model (9), (10), (15) and (16). The following assumptions are needed in the derivation of the main results.

Assumption 1: The initial states $x_i(t_{i,0}) = x(T_0)$ are uncorrelated to $\omega(T_k)$ and $\upsilon_i(T_k)$, and $\mathbf{E}\{x(T_0)\} = x_0$, $\mathbf{E}\{(x(T_0)-x_0)(x(T_0)-x_0)^{\mathrm{T}}\} = P_0$. $\omega(T_k)$ is uncorrelated to $\upsilon_i(T_k)$. $\upsilon_r(T_k)$ is correlated to $\upsilon_s(T_k)$, and the covariance is given by $\mathbf{E}\{\upsilon_r(T_k)\upsilon_s^{\mathrm{T}}(T_j)\} = Q_{r,s}\delta_{kj}$, $i \in Z_0$.

Assumption 2: $\alpha_i(t_{i,k})$ are mutually independent and are independent of $x_i(t_{i,0})$, $\omega(T_k)$ and $\upsilon_i(T_k)$, $i \in Z_0$.

III. Design of Fusion Estimators

A. Design of Local Estimators

Define $\Theta_i(t_{i,k}) = \mathbf{E}\{x_i(t_{i,k})x_i^{\mathrm{T}}(t_{i,k})\}$, $i \in Z_0$, then by the fact $x_i(t_{i,k}) \perp \omega_i(t_{i,k})$ one has that

$$\begin{aligned} \Theta_i(t_{i,k+1}) &= \alpha_i(1-\alpha_i)(A_1 - A_2)\Theta_i(t_{i,k})(A_1 - A_2)^{\mathrm{T}} \\ &\quad + \bar{A}_i \Theta_i(t_{i,k}) \bar{A}_i^{\mathrm{T}} + Q_{\omega_i} \end{aligned} \tag{18}$$

where $\bar{A}_i = \mathbf{E}\{\tilde{A}_i(t_{i,k})\} = \alpha_i A_1 + (1-\alpha_i) A_2$. From the distribution of $\alpha_i(t_{i,k})$, one has

$$\mathbf{E}\{(\alpha_i(t_{i,k}) - \alpha_i)^2\} = \alpha_i(1-\alpha_i) \tag{19}$$

$$\mathbf{E}\{(\alpha_i(t_{i,k}) - \alpha_i)(\alpha_j(t_{i,k}) - \alpha_j)\} = 0,\ i \neq j \tag{20}$$

The optimal local estimator for sensor i, $i \in Z_0$ is given in the following theorem.

Theorem 1: For sensor i with a nonuniform estimating rate $h_i(t_{i,k})$ satisfying (13) and (14), the local recursive optimal linear estimator is given by

$$\hat{x}_i(t_{i,k+1}|t_{i,k}) = \bar{A}_i \hat{x}_i(t_{i,k}|t_{i,k}) \tag{21}$$

$$\begin{aligned} \hat{x}_i(t_{i,k+1}|t_{i,k+1}) &= \hat{x}_i(t_{i,k+1}|t_{i,k}) \\ &\quad + K_i(t_{i,k+1})\varepsilon_i(t_{i,k+1}) \end{aligned} \tag{22}$$

$$\varepsilon_i(t_{i,k+1}) = y_i(t_{i,k+1}) - C_i \hat{x}_i(t_{i,k+1}|t_{i,k}) \tag{23}$$

$$K_i(t_{i,k+1}) = P_i(t_{i,k+1}|t_{i,k}) C_i^{\mathrm{T}} \Omega_i^{-1}(t_{i,k+1}) \tag{24}$$

$$\Omega_i(t_{i,k+1}) = C_i P_i(t_{i,k+1}|t_{i,k}) C_i^{\mathrm{T}} + D_i Q_i D_i^{\mathrm{T}} \tag{25}$$

$$\begin{aligned} P_i(t_{i,k+1}|t_{i,k}) &= \bar{A}_i P_i(t_{i,k}|t_{i,k}) \bar{A}_i^{\mathrm{T}} + Q_{\omega_i} + \\ &\quad \alpha_i(1-\alpha_i)(A_1 - A_2)\Theta_i(t_{i,k})(A_1 - A_2)^{\mathrm{T}} \end{aligned} \tag{26}$$

$$\begin{aligned} &P_i(t_{i,k+1}|t_{i,k+1}) = (I - K_i(t_{i,k+1})C_i) P_i(t_{i,k+1}|t_{i,k}) \times \\ &(I - K_i(t_{i,k+1})C_i)^{\mathrm{T}} + K_i(t_{i,k+1}) D_i Q_i D_i^{\mathrm{T}} K_i^{\mathrm{T}}(t_{i,k+1}) \end{aligned} \tag{27}$$

where $\varepsilon_i(t_{i,k}) = y_i(t_{i,k}) - \hat{y}_i(t_{i,k}|t_{i,k-1})$ is the innovation, $\Omega_i(t_{i,k}) = \mathbf{E}\{\varepsilon_i(t_{i,k})\varepsilon_i^{\mathrm{T}}(t_{i,k})\}$, $\hat{x}_i(t_{i,0}|t_{i,0}) = x_0$, $P_i(t_{i,0}|t_{i,0}) = P_0$.

Proof: The proof can be followed by applying the projection principle and innovation analysis [11], and is omitted here for brevity.

Remark 1: When $\alpha_i(t_{i,k}) = 0$ or $\alpha_i(t_{i,k}) = 1$, i.e., the estimation rates of the sensors are uniform and constant, then $\bar{A}_i$ reduces to A_1 or A_2, and the optimal estimator given in Theorem 1 reduces to the standard Kalman estimator.

B. Design of the Fusion Rule

The considered fusion estimation consists of two steps. At the first step, each sensor generates local estimates by applying the algorithm in Theorem 1. At the second step, each sensor collects local estimates from itself and the other sensors to generate fused estimates. Since the sensors generates local estimates asynchronously, local estimates from the other sensors may not be available for fusion estimation at each of the sensors. Take the sensor r for example, at the estimating instants $t_{r,k}$, it collects local estimates $\hat{x}_r(t_{r,k}|t_{r,k})$ and $\hat{x}_s(t_{r,k}|t_{r,k})$ to generate a fused estimate $\hat{x}_{or}(t_{r,k}|t_{r,k})$ according to a fusing rule to be designed if $\hat{x}_s(t_{r,k}|t_{r,k})$ from sensor s is available at instant $t_{r,k}$ (note that $t_{r,k} = t_{s,k}$ in this case). Otherwise, it keeps its own local estimate $\hat{x}_r(t_{r,k}|t_{r,k})$ as the fused one.

In what follows, a fusion rule for sensor r will be designed, and the results for sensor s can be obtained by following the similar lines. For notational convenience, we will write h_i for $h_i(t_{i,k})$, $i \in Z_0$ in the following development. Denote by t_k^o the time instants when $\hat{x}_s$ is available from sensor s for generating fused estimates at sensor r, and denote by t_k^c

the time instants when $\hat{x}_s$ is not available. Then, one has

$$\{t_{r,0}, t_{r,1}, \ldots\} = \{t_0^o, t_1^o, \ldots\} \bigcup \{t_0^c, t_1^c, \ldots\} \tag{28}$$

Lemma 1: [4] Let $\hat{x}_i$, $i \in \bar{Z} = \{1, \ldots, \bar{m}\}$ be unbiased estimates of a stochastic vector $x \in \Re^n$. Assume that the errors $\tilde{x}_i = x - \hat{x}_i$ are mutually correlated, then the optimal fusion estimate of x with matrix weights is given by

$$\hat{x}_o = \sum_{i=1}^{\bar{m}} A_{oi}\hat{x}_i \tag{29}$$

where the matrix weights A_{oi} are computed by $\text{col}\{A_{oi}^{\mathrm{T}}\}_{i\in\bar{Z}} = \Lambda^{-1}e(e^{\mathrm{T}}\Lambda^{-1}e)^{-1}$, $\Lambda = [P_{ij}]$, $i,j \in \bar{Z}$, P_{ii} is the variance of $\tilde{x}_i$, P_{ij} is the covariance of $\tilde{x}_i$ and $\tilde{x}_j$, $i \neq j$, and $e = [\underbrace{I, \ldots, I}_{\bar{m}}]^{\mathrm{T}}$. The corresponding variance matrix of the fused estimation error is given by $P_o = (e^{\mathrm{T}}\Lambda^{-1}e)^{-1}$, and one has that $P_o \leq P_{ii}$, $i \in \bar{Z}$.

The optimal fusion rule with matrix weights in sensor r is given in the following theorem.

Theorem 2: For the system (9), (10), (15) and (16) satisfying Assumptions 1 and 2, the fusion estimator in the sensor r is given by

$$\hat{x}_{or}(t_{r,k}|t_{r,k}) = \begin{cases} \hat{x}_r(t_l^c|t_l^c), \ t_{r,k} \in \{t_l^c, l = 0,1,2,\ldots\} \\ A_{or}(t_l^o)\hat{x}_r(t_l^o|t_l^o) + A_{os}(t_l^o)\hat{x}_s(t_l^o|t_l^o), \\ \quad t_{r,k} \in \{t_l^o, l = 0,1,2,\ldots\} \end{cases} \tag{30}$$

The corresponding variance matrix of the fusion estimation error is computed by

$$P_{or}(t_{r,k}|t_{r,k}) = \begin{cases} P_r(t_l^c|t_l^c), \ t_{r,k} \in \{t_l^c, l = 0,1,2,\ldots\} \\ \bar{P}_{or}(t_l^o|t_l^o), t_{r,k} \in \{t_l^o, l = 0,1,2,\ldots\} \end{cases} \tag{31}$$

$$\bar{P}_{or}(t_l^o|t_l^o) = (e^{\mathrm{T}}\Delta_r^{-1}(t_l^o)e)^{-1} \tag{32}$$

and one has $P_{or}(t_{r,k}|t_{r,k}) \leq P_r(t_{r,k}|t_{r,k})$, where the optimal matrix weights $A_{or}(t_l^o)$ and $A_{os}(t_l^o)$ are computed by

$$\begin{bmatrix} A_{or}^{\mathrm{T}}(t_l^o) \\ A_{os}^{\mathrm{T}}(t_l^o) \end{bmatrix} = \Delta_r^{-1}(t_l^o)e(e^{\mathrm{T}}\Delta_r^{-1}(t_l^o)e)^{-1} \tag{33}$$

$$\Delta_r(t_l^o) = \begin{bmatrix} P_r(t_l^o|t_l^o) & P_{r,s}(t_l^o|t_l^o) \\ P_{r,s}^{\mathrm{T}}(t_l^o|t_l^o) & P_s(t_l^o|t_l^o) \end{bmatrix}, \ e = [I \quad I]^{\mathrm{T}} \tag{34}$$

$\hat{x}_r(t_{r,k}|t_{r,k})$, $\hat{x}_s(t_{r,k}|t_{r,k})$, $P_r(t_l^o|t_l^o)$ and $P_s(t_l^o|t_l^o)$ are computed by Theorem 1, $P_{r,s}(t_l^o|t_l^o)$ is the covariance matrix of estimation errors in sensors r and s.

Proof: (30)-(34) can be followed by (28) and Lemma 1. Moreover, by Lemma 1 one has that $\bar{P}_{or}(t_l^o|t_l^o) \leq P_r(t_l^o|t_l^o)$. Thus, it follows from (31) that $P_{or}(t_{r,k}|t_{r,k}) \leq P_r(t_{r,k}|t_{r,k})$.

It can be seen from (34) that the computation of the covariance matrix $P_{r,s}(t_l^o|t_l^o)$ is one of the key issues in applying the fusion estimator in Theorem 2. Denote by $n_i(t_k^o)$ the number of sampling intervals of sensor i over the interval $[t_k^o, t_{k+1}^o]$, i.e., $t_{k+1}^o - t_k^o = n_i(t_k^o)h_i$, $i \in \{r, s\}$. Let $\psi_1 = [t_k^o + lh_r, t_k^o + lh_r + \tilde{a}_r(t_k^o + lh_r)h]$ and $\psi_2 = [t_k^o + qh_s, t_k^o + qh_s + \tilde{a}_s(t_k^o + qh_s)h]$, where $l \in \{0, 1, \ldots, n_r(t_k^o)-1\}$ and $q \in \{0, 1, \ldots, n_s(t_k^o)-1\}$. ψ_1 and ψ_2 represent, respectively, the sampling intervals of sensor r and sensor s over the interval $[t_k^o, t_{k+1}^o]$. If $\psi_1 \bigcap \psi_2 \neq \phi$, then denote

$$\begin{aligned} \psi_1 \bigcap \psi_2 = [&t_k^o + lh_r + u_{r,l}(t_k^o)h, \ t_k^o + lh_r + (u_{r,l}(t_k^o) \\ &+ m_{l,q}(t_k^o))h] = [t_k^o + qh_s + u_{s,q}(t_k^o)h, \ t_k^o \\ &+ qh_s + (u_{s,q}(t_k^o) + m_{l,q}(t_k^o))h] \end{aligned} \tag{35}$$

where $m_{l,q}(t_k^o)h$ is the overlap interval of the two time intervals ψ_1 and ψ_2. Then, a recursive equation for computing $P_{r,s}(t_l^o|t_l^o)$ is given in the following theorem.

Theorem 3: For the system (9), (10), (15) and (16) satisfying Assumptions 1 and 2, the covariance of local estimation errors in sensors r and s satisfies the recursive equation

$$P_{r,s}(t_{k+1}^o|t_{k+1}^o) = \sum_{i=1}^{3} \chi_i \tag{36}$$

where

$$\begin{aligned}
\chi_1 = &\prod_{l=1}^{n_r(t_k^o)} (\bar{A}_r - K_r(t_k^o + lh_r)C_r\bar{A}_r)P_{r,s}(t_k^o|t_k^o) \times \\
&\left[\prod_{l=1}^{n_s(t_k^o)} (\bar{A}_s - K_s(t_k^o + lh_s)C_s\bar{A}_s)\right]^{\mathrm{T}} \\
\chi_2 = &\sum_{l=0}^{n_r(t_k^o)-1} \sum_{q=0}^{n_s(t_k^o)-1} \prod_{j=l+2}^{n_r(t_k^o)} (\bar{A}_r - K_r(t_k^o + jh_r)C_r\bar{A}_r) \times \\
&(I - K_r(t_k^o + (l+1)h_r)C_r)\Upsilon_{l,q}(t_k^o) \times \\
&(I - K_s(t_k^o + (l+1)h_s)C_s)^{\mathrm{T}} \times \\
&\left[\prod_{j=q+2}^{n_s(t_k^o)} (\bar{A}_s - K_s(t_k^o + jh_s)C_s\bar{A}_s)\right]^{\mathrm{T}} \\
\chi_3 = &K_r(t_{k+1}^o)D_rQ_{rs}D_r^{\mathrm{T}}K_s^{\mathrm{T}}(t_{k+1}^o) \\
\Upsilon_{l,q}(t_k^o) = &\begin{cases} 0, & \psi_1 \bigcap \psi_2 = \phi \\ \bar{\Upsilon}_{l,q}(t_k^o), & \psi_1 \bigcap \psi_2 \neq \phi \end{cases} \\
\bar{\Upsilon}_{l,q}(t_k^o) = &\alpha_r\alpha_s\Pi_{1,1}(t_k^o) + \alpha_r(1-\alpha_s)\Pi_{1,2}(t_k^o) \\
&+\alpha_s(1-\alpha_r)\Pi_{1,2}^{\mathrm{T}}(t_k^o) + (1-\alpha_r)(1-\alpha_s)\Pi_{2,2}(t_k^o) \\
\Pi_{i,j} = &\sum_{j=1}^{m_{l,q}(t_k^o)+1} A^{a_i - u_{r,l}(t_k^o) - j}BQ_\omega \times \\
&B^{\mathrm{T}}\left(A^{a_j - u_{s,q}(t_k^o) - j}\right)^{\mathrm{T}}, \ i,j = 1,2
\end{aligned}$$

Proof: Subtracting $x_i(t_{i,k+1})$ from both sides of (21) and taking the state equation in (9) into account yield

$$\begin{aligned} \tilde{x}_i(t_{i,k+1}|t_{i,k}) = &\bar{A}_i\tilde{x}_i(t_{i,k}|t_{i,k}) + \omega_i(t_{i,k}) \\ &+(\alpha_i(t_{i,k}) - \alpha_i)(A_1 - A_2)x_i(t_{i,k}) \end{aligned} \tag{37}$$

Subtracting $x_i(t_{i,k+1})$ from both sides of (22) leads to

$$\begin{aligned} \tilde{x}_i(t_{i,k+1}|t_{i,k+1}) \quad = \quad &(I - K_i(t_{i,k+1})\bar{C}_i)\tilde{x}_i(t_{i,k+1}|t_{i,k}) \\ &-K_i(t_{i,k+1})D_i\upsilon_i(t_{i,k+1}) \end{aligned} \tag{38}$$

Substituting (37) into (38) leads to

$$\begin{aligned}&\tilde{x}_i(t_{i,k+1}|t_{i,k+1}) = (I - K_i(t_{i,k+1})C_i)\bar{A}_i\tilde{x}_i(t_{i,k}|t_{i,k})\\&+(\alpha_i(t_{i,k}) - \alpha_i)(I - K_i(t_{i,k+1})C_i)(A_1 - A_2)x_i(t_{i,k})\\&+(I - K_i(t_{i,k+1})C_i)\omega_i(t_{i,k}) - K_i(t_{i,k+1})D_i\upsilon_i(t_{i,k+1}) \quad (39)\end{aligned}$$

Applying (39) recursively, one obtains the following estimation errors at the time scale t_k^o

$$\tilde{x}_i(t_{k+1}^o|t_{k+1}^o) = \sum_{j=1}^{3}\eta_{i,j}(t_k^o) - \eta_{i,4}(t_k^o),\ i \in \{r,s\} \quad (40)$$

where

$$\begin{aligned}\eta_{i,1}(t_k^o) =& \prod_{l=1}^{n_i(t_k^o)}(\bar{A}_i - K_i(t_k^o + lh_i)C_i\bar{A}_i)\tilde{x}_i(t_k^o|t_k^o)\\ \eta_{i,2}(t_k^o) =& \sum_{l=0}^{n_i(t_k^o)-1}(\alpha_i(t_k^o + lh_i) - \alpha_i)\times\\&\prod_{j=l+2}^{n_i(t_k^o)}(\bar{A}_i - K_i(t_k^o + jh_i)C_i\bar{A}_i)\times\\&(I - K_i(t_k^o + (l+1)h_i)C_i)(A_1 - A_2)x_i(t_k^o + lh_r)\\ \eta_{i,3}(t_k^o) =& \sum_{l=0}^{n_i(t_k^o)-1}\prod_{j=l+2}^{n_i(t_k^o)}(\bar{A}_i - K_i(t_k^o + jh_i)C_i\bar{A}_i)\times\\&(I - K_i(t_k^o + (l+1)h_i)C_i)\omega_i(t_k^o + lh_i)\\ \eta_{i,4}(t_k^o) =& \sum_{l=1}^{n_i(t_k^o)}\prod_{j=l+1}^{n_i(t_k^o)}(\bar{A}_i - K_i(t_k^o + jh_i)C_i\bar{A}_i)\times\\&K_i(t_k^o + lh_i)D_i\upsilon_i(t_k^o + lh_i)\end{aligned}$$

and we define $\prod_{j=a}^{b} f(j) = I$ if $b < a$. Since $\mathbf{E}\{\alpha_i(t_k^o+lh_i) - \alpha_i\} = 0$, $\tilde{x}_r(t_k^o|t_k^o)\perp\omega_s(t_k^o+lh_s)$ and $\tilde{x}_r(t_k^o|t_k^o)\perp\upsilon_s(t_k^o+lh_s)$, $l = 0, 1, \ldots, n_s(t_k^o)$, by (40) one has that

$$\begin{aligned}P_{r,s}(t_{k+1}^o|t_{k+1}^o) &= \mathbf{E}\{\tilde{x}_r(t_{k+1}^o|t_{k+1}^o)\tilde{x}_s^{\mathrm{T}}(t_{k+1}^o|t_{k+1}^o)\}\\&= \sum_{j=1}^{4}\mathbf{E}\{\eta_{r,j}(t_k^o)\eta_{s,j}^{\mathrm{T}}(t_k^o)\} \quad (41)\end{aligned}$$

By following some routine computation, one has

$$\mathbf{E}\{\eta_{r,1}(t_k^o)\eta_{s,1}^{\mathrm{T}}(t_k^o)\} = \chi_1 \quad (42)$$

Since $t_k^o + lh_r \neq t_k^o + qh_s$, $\forall\ l \in \{1, \ldots, n_r(t_k^o) - 1\}$, $q \in \{1, \ldots, n_s(t_k^o) - 1\}$, by noting $t_k^o + n_i(t_k^o)h_i = t_{k+1}^o$, $i \in \{r, s\}$ one has that

$$\mathbf{E}\{\eta_{r,4}(t_k^o)\eta_{s,4}^{\mathrm{T}}(t_k^o)\} = \chi_3 \quad (43)$$

It follows from (20) that

$$\mathbf{E}\{\eta_{r,2}(t_k^o)\eta_{s,2}^{\mathrm{T}}(t_k^o)\} = 0 \quad (44)$$

On the other hand, one has

$$\mathbf{E}\{\eta_{r,3}(t_k^o)\eta_{s,3}^{\mathrm{T}}(t_k^o)\} = \chi_2 \quad (45)$$

where $\Upsilon_{l,q}(t_k^o) = \mathbf{E}\{\omega_r(t_k^o + lh_r)\omega_s^{\mathrm{T}}(t_k^o + qh_s)\}$, $l \in \{0, 1, \ldots, n_r(t_k^o) - 1\}$, $q \in \{0, 1, \ldots, n_s(t_k^o) - 1\}$. If $\psi_1 \bigcap \psi_2 = \phi$, then one has $\Upsilon_{l,q}(t_k^o) = 0$ by the fact that $\omega_i(t_{i,k})$ are white noises. If $\psi_1 \bigcap \psi_2 \neq \phi$, then by (11), (15), (16) and (35) one has that

$$\begin{aligned}\Upsilon_{l,q}(t_k^o) =& \mathbf{E}\Bigg\{\Bigg[\alpha_r(t_k^o + lh_r)\sum_{j=0}^{a_1-1}\vartheta_{r,1}(j)\\&+(1 - \alpha_r(t_k^o + lh_r))\sum_{j=0}^{a_2-1}\vartheta_{r,2}(j)\Bigg]\cdot\\&\Bigg[\alpha_s(t_k^o + qh_s)\sum_{j=0}^{a_1-1}\vartheta_{s,1}(j)\\&+(1 - \alpha_s(t_k^o + qh_s))\sum_{j=0}^{a_2-1}\vartheta_{s,2}(j)\Bigg]^{\mathrm{T}}\Bigg\}\\ =& \mathbf{E}\Bigg\{\Bigg[\alpha_r(t_k^o + lh_r)\sum_{j=u_{r,l}(t_k^o)}^{u_{r,l}(t_k^o)+m_{l,q}(t_k^o)}\vartheta_{r,1}(j)\\&+(1 - \alpha_r(t_k^o + lh_r))\sum_{j=u_{r,l}(t_k^o)}^{u_{r,l}(t_k^o)+m_{l,q}(t_k^o)}\vartheta_{r,2}(j)\Bigg]\cdot\\&\Bigg[\alpha_s(t_k^o + qh_s)\sum_{j=u_{s,q}(t_k^o)}^{u_{s,q}(t_k^o)+m_{l,q}(t_k^o)}\vartheta_{s,1}(j)\\&+(1 - \alpha_s(t_k^o + qh_s))\sum_{j=u_{s,q}(t_k^o)}^{u_{s,q}(t_k^o)+m_{l,q}(t_k^o)}\vartheta_{s,2}(j)\Bigg]^{\mathrm{T}}\Bigg\}\\ =& \bar{\Upsilon}_{l,q}(t_k^o) \quad (46)\end{aligned}$$

where $\vartheta_{r,\kappa}(j) = A^{a_\kappa-j-1}B\omega(t_k^o + lh_r + jh)$ and $\vartheta_{s,\kappa}(j) = A^{a_\kappa-j-1}B\omega(t_k^o + qh_s + jh)$, $\kappa = 1, 2$. Then, (36) follows from (41)-(46).

Remark 2: By setting the error covariances in Theorem 3 to zero, the fusion estimation algorithm given in Theorems 1-3 is also applicable to the case where the measurement noises are mutually uncorrelated.

IV. AN ILLUSTRATIVE EXAMPLE

An example of maneuvering target tracking system is presented, where the target's position and velocity evolve according to the model in (1) with

$$A = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, B = \sqrt{10}\begin{bmatrix} h^2/2 \\ h \end{bmatrix} \quad (47)$$

where h is the sampling period and is assumed to be $0.5s$. The state is $x(T_k) = [x_p(T_k)\ \ x_v(T_k)]^{\mathrm{T}}$, where $x_p(T_k)$ and $x_v(T_k)$ are the position and velocity of the target at time T_k.

Two sensors, namely, sensors r and s, are deployed to monitor the outputs of the system (47). Suppose that the measurements from the two sensors are generated non-uniformly with two rates $h_1 = h$ and $h_2 = 3h$, and the measurement generation rates in sensors r and s vary between the two rates according to white Bernoulli processes $\alpha_r(t_{r,k})$ and $\alpha_s(t_{s,k})$, respectively, where $\mathbf{E}\{\alpha_r(t_{r,k})\} = \alpha_r = 0.5$ and

$\mathbf{E}\{\alpha_s(t_{s,k})\} = \alpha_s = 0.6$, i.e., the measurement generation rate in sensor r takes h_1 and h_2 with probabilities 0.5 and 0.5, respectively, and the measurement generation rate in sensor s takes h_1 and h_2 with probabilities 0.6 and 0.4, respectively. The trajectories of $\alpha_r(t_{r,k})$ and $\alpha_s(t_{s,k})$ are depicted in Fig. 2. Then, the output equations in the two sensors are given by (3) with $C_r = [0.8 \;\; 0]$, $C_s = [0.3 \;\; 0]$, $D_r = 0.5$ and $D_s = 0.7$. Assume that $Q_\omega = 1.0$, $Q_r = 1.5$, $Q_s = 1.7$, and $Q_{rs} = 1.0$.

Firstly, each sensor generates local estimates when the measurements are available. Secondly, each sensor generates fused estimates by fusing local estimates from the two sensors if the local estimates from another sensor is available. Otherwise, it keeps its own estimates as the fused ones. The simulations are shown in Figs. 3-6, where Fig.3 shows the true values and estimates of the position, while Fig. 4 shows the true values and estimates of the velocity, Fig. 5 depicts the traces of the local estimation error variances in sensor r and sensor s and the fused estimation error variance in sensor r, while Fig. 6 depicts the traces of the local estimation error variances in sensor r and sensor s and the fused estimation error variance in sensor s. It can be seen from Figs. 5 and 6 that the estimation performance of each sensor is improved by fusing local estimates from the two sensors, showing the effectiveness of the proposed fusion estimator design.

V. CONCLUSION

A fusion estimation algorithm has been designed in this paper for two-sensor stochastic systems with non-uniform estimation rates. The algorithm is also applicable to fusion estimation systems where the sensors may not be time-synchronized. Extending the results to multi-sensor case remains to be our future work.

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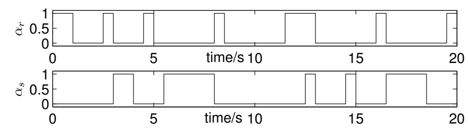

Fig. 2. Trajectories of $\alpha_r(t_{r,k})$ and $\alpha_s(t_{s,k})$.

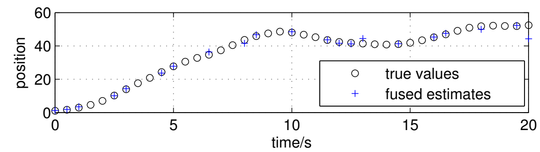

Fig. 3. True values and estimates of the positions in sensor r.

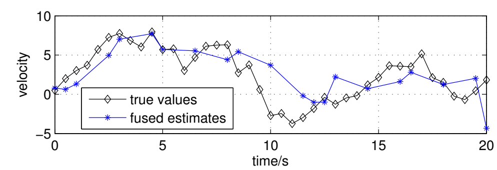

Fig. 4. True values and estimates of the velocities in sensor r.

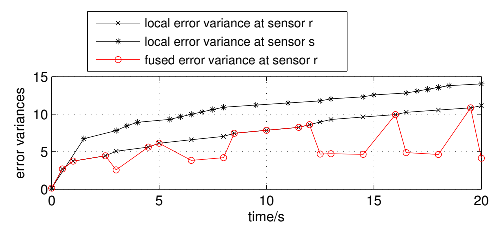

Fig. 5. $P_r(t_{r,k}|t_{r,k})$, $P_s(t_{s,k}|t_{s,k})$ and $P_{or}(t_{r,k}|t_{r,k})$.

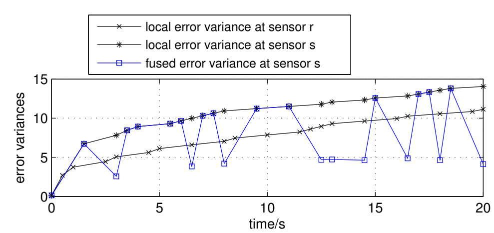

Fig. 6. $P_r(t_{r,k}|t_{r,k})$, $P_s(t_{s,k}|t_{s,k})$ and $P_{os}(t_{s,k}|t_{s,k})$.

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